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MONSTROUS MOONSHINE AND THE UNIQUENESS OF THE MOONSHINE MODULE *

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ABSTRACT

In this talk we consider the relationship between the conjectured uniqueness of the Moonshine module $\mathcal{V}^\natural$ of Frenkel, Lepowsky and Meurman and Monstrous Moonshine, the genus zero property for Thompson series discovered by Conway and Norton. We discuss some evidence to support the uniqueness of $\mathcal{V}^\natural$ by considering possible alternative orbifold constructions of $\mathcal{V}^\natural$ from a Leech lattice compactified string. Within these constructions we find a new relationship between the centralisers of the Monster group and the Conway group generalising an observation made by Conway and Norton. We also relate the uniqueness of $\mathcal{V}^\natural$ to Monstrous Moonshine and argue that given this uniqueness, then the genus zero properties hold if and only if orbifolding $\mathcal{V}^\natural$ with respect to a monster element reproduces $\mathcal{V}^\natural$ or the Leech theory.

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The Moonshine Module. The Moonshine module [1] of Frenkel, Lepowsky and Meurman (FLM) is the first example of an orbifold CFT [2] and is constructed from a string compactified to R^{24}/Λ where Λ is the Leech lattice, the unique 24 dimensional even self-dual lattice without roots i.e. $\lambda^2 \neq 2$ cf. [3]. The orbifolding is then based on the Z_2 reflection automorphism of Λ.

Let $\mathcal{V}^\Lambda$ denote the set of vertex operators $\{\phi(z)\}$ for the Leech lattice CFT which forms a closed meromorphic operator product algebra (OPA) with central charge 24 [1,4]

$$\phi_i(z)\phi_j(w) \sim \sum_k C_{ijk}(z-w)^{h_k-h_i-h_j}\phi_k(w) + \ldots \tag{1}$$

We will represent such an OPA schematically by $\phi\phi \sim \phi$. The 1-loop partition function $Z(\tau) = \mathrm{Tr}_{\mathcal{V}^\Lambda}(q^{L_0})$ is a modular invariant and meromorphic function of τ with a unique simple pole at $q = e^{2\pi i\tau} = 0$ and is given by the unique (up to an additive constant) modular invariant function $J(\tau)$

$$\begin{aligned} Z(\tau) &= J(\tau) + 24 \\ J(\tau) &= \frac{E_2^3}{\eta^{24}} - 744 = \frac{1}{q} + 0 + 196884q + \ldots \end{aligned} \quad . \tag{2}$$

The constant 24 reflects the existence of 24 massless (conformal weight 1) operators in this theory. $\eta(\tau) = q^{1/24}\prod_n(1-q^n)$ and $E_2(\tau)$ is the Eisenstein modular form of weight 4 [5].

The FLM Moonshine module [1] is an orbifold CFT based on the Z_2 lattice reflection automorphism $\bar{r} : \lambda \to -\lambda$ for $\lambda \in \Lambda$. $\bar{r}$ lifts to a family of Z_2 automorphisms of $\mathcal{V}^\Lambda$ preserving (1) from which family one automorphism r is chosen. Defining the projection $\mathcal{P}_r = (1+r)/2$, the set of operators $\mathcal{P}_r\mathcal{V}^\Lambda$ then also form a closed meromorphic OPA. However, the corresponding partition function $\mathrm{Tr}_{\mathcal{P}_r\mathcal{V}^\Lambda}(q^{L_0}) = \frac{1}{2}({}^{1}\underset{1}{\Box} + {}^{r}\underset{1}{\Box})$ is not modular invariant, employing standard notation for the world-sheet torus boundary conditions e.g. [6]. Thus, under a modular transformation $\tau \to -1/\tau$, ${}^{r}\underset{1}{\Box} = 1/\eta_{\bar{r}}(\tau) \to {}^{1}\underset{r}{\Box} = 2^{12}\eta_{\bar{r}}(\tau/2) = 2^{12}q^{1/2} + \ldots$ where $\eta_{\bar{r}}(\tau) = [\eta(2\tau)/\eta(\tau)]^{24}$. Therefore the introduction of a 'twisted' sector with vacuum energy 1/2 and degeneracy 2^{12} is necessary to form a modular invariant theory [1,2]. The states of this sector are constructed from twisted vertex operators $\mathcal{V}_r = \{\psi(z)\}$ acting on the untwisted vacuum. Thus $\mathcal{V}^\Lambda$ is enlarged by $\mathcal{V}_r$ to $\mathcal{V}' = \mathcal{V}^\Lambda \oplus \mathcal{V}_r$ which forms a closed non-meromorphic OPA [1,7,8,9] where (schematically)

$$\phi\phi \sim \phi, \qquad \phi\psi \sim \psi, \qquad \psi\psi \sim \phi \tag{3}$$

$\bar{r}$ can also be lifted to an automorphism r of (3) where the operators of $\mathcal{P}_r\mathcal{V}_r$ have integral conformal weight. Then $\mathcal{V}^\natural = \mathcal{P}_r\mathcal{V}'$ forms a closed meromorphic OPA, the FLM Moonshine module [1]. The r projection ensures the absence of untwisted massless operators whereas the twisted sector operators are all massive since the twisted vacuum energy is 1/2. Thus the orbifold partition function is

$$\mathrm{Tr}_{\mathcal{V}^\natural}(q^{L_0}) = \mathcal{P}_r \underset{1}{\square} + \mathcal{P}_r \underset{r}{\square} = J(\tau) \tag{4}$$

The absence of massless operators in $\mathcal{V}^\natural$ sets the Moonshine module apart from other CFTs. Usually such operators are present and form a closed Kac-Moody algebra. However, the 196884 conformal weight 2 operators in $\mathcal{V}^\natural$, including the energy-momentum tensor $T(z)$ can be used to define a closed non-associative commutative algebra. FLM demonstrated [1] that this algebra is an affine version of the 196883 dimensional Griess algebra [10] together with $T(z)$. The automorphism group of the Griess algebra is the Monster M. FLM showed that M is the automorphism group for the OPA of $\mathcal{V}^\natural$ where the operators of $\mathcal{V}^\natural$ of a given conformal weight form a (reducible) representation of M. This demonstrates an observation of McKay and Thompson [11] that the coefficients of $J(\tau)$ are positive sums of dimensions of irreducible representations of M e.g. the coefficient of q is $196884 = 1 + 196883$, the sum of the trivial and adjoint representation.

We may identify an involution $i \in M$, defined like a 'fermion number', under which all untwisted (twisted) operators have eigenvalue $+1(-1)$ where i also respects (3). The centraliser of i can be found [1] to give $C(i|M) = \{g \in M | ig = gi\} = 2_+^{1+24}.\mathrm{Co}_1$ where Co_1 is the Conway simple group (the automorphism group Co_0 of Λ modulo the reflection automorphism $\bar{r}$), 2_+^{1+24} is an extra-special group and $A.B$ denotes a group with normal subgroup A with $B = A.B/A$. This result is an essential part of the FLM construction since M is generated by $2_+^{1+24}.\mathrm{Co}_1$ and a second involution σ [10]. FLM constructed σ, which mixes the untwisted and twisted sectors, from a hidden triality symmetry [1,12] and hence showed that the automorphism group of $\mathcal{V}^\natural$ is M.

The automorphisms i and r can be said to be 'dual' to each other in the sense that they are both automorphisms of $\mathcal{V}'$ and that the subsets invariant under i and r, $\mathcal{V}^\Lambda$ and $\mathcal{V}^\natural$ repectively, form meromorphic OPAs. In addition, we may 'reorbifold' $\mathcal{V}^\natural$ with respect to i to reproduce $\mathcal{V}^\Lambda$. Thus

$$\begin{array}{ccc} & \mathcal{V}' & \\ \mathcal{P}_i \swarrow & & \searrow \mathcal{P}_r \\ \mathcal{V}^\Lambda & \underset{i}{\overset{r}{\rightleftarrows}} & \mathcal{V}^\natural \end{array} \tag{5}$$

where the horizontal arrows denote an orbifolding and the diagonal arrows a projection [13].

Monstrous Moonshine. The operators of $\mathcal{V}^\natural$ of a given conformal weight form reducible representations of M. The Thompson series $T_g(\tau)$ for $g \in M$ is defined by the trace

$$T_g(\tau) = \mathrm{Tr}_{\mathcal{V}^\natural}(gq^{L_0}) = \frac{1}{q} + 0 + [1 + \chi(g)]q + \ldots \tag{6}$$

which depends only on the conjugacy class of g where $\chi(g)$ is the character in the 196883 dimensional irreducible representation. Thus for i defined above, it is easy to show $T_i(\tau) = [\eta_{\overline{T}}(\tau)]^{-1} + 24$.

The Thompson series for the identity element is $J(\tau)$ which is unique (up to a constant) for the following reasons. Let $\mathcal{F} = H/\Gamma$ be the fundamental region where $\Gamma = \mathrm{SL}(2, Z)$ is the full modular group and H is the upper half complex plane. Adding the point at infinity, the compactification $\overline{\mathcal{F}}$ is isomorphic to the Riemann sphere of genus zero where the function $J(\tau)$ realises this isomorphism. Such a function is called a *hauptmodul for the genus zero modular group* Γ. A modular invariant meromorphic function is a hauptmodul if and only if it possesses a unique simple pole. Once the location of this pole is specified, this function is itself unique up to a constant cf. [5,14].

Based on 'experimental' evidence, Conway and Norton [15] conjectured that each $T_g(\tau)$ is a hauptmodul for a genus zero modular group Γ_g. This has recently been rigorously demonstrated by Borcherds although the origin of the genus zero property remains obscure [16]. In general, for g of order n, $T_g(\tau)$ is found to be invariant up to phases of order (at most) h under $\Gamma_0(n) = \{\begin{pmatrix} a & b \\ nc & d \end{pmatrix} | \det = 1\}$ where $h|n$ and $h|24$. $T_g(\tau)$ is fixed by Γ_g with $\Gamma_0(N) \subseteq \Gamma_g \subseteq \mathcal{N}(N) = \{\rho \in \mathrm{SL}(2, R) | \rho\Gamma_0(N) = \Gamma_0(N)\rho\}$, the normaliser of $\Gamma_0(N)$ in $\mathrm{SL}(2, R)$ where $N = nh$. Furthermore, Γ_g is a genus zero modular group and $T_g(\tau)$ is the corresponding hauptmodul with a simple pole at $q = 0$. Consider the elements of prime order $n = p$. Apart from one class of order 3 with $h = 3$, we have $h = 1$ in each case. Thus either $\Gamma_g = \Gamma_0(p)$ or $\Gamma_0(p)+$, generated by $\Gamma_0(p)$ and the Fricke involution $W_p : \tau \to -1/p\tau$ with $W_p^2 = 1$, the only non-trivial element of $\mathcal{N}(p)$. $\Gamma_0(p)$ is of genus zero when $(p-1)|24$ $(p = 2, 3, 5, 7, 13)$ where the hauptmodul is $[\eta(\tau)/\eta(p\tau)]^{2d} + 2d$ with $2d = 24/(p-1)$. There is a class of M denoted by $p-$ for each such prime with this Thompson series e.g. the involution i belongs to the class $2-$. $\Gamma_0(p)+$ is of genus zero for $2 \leq p \leq 31$ or $p = 41, 47, 59, 71$, which constitute all the prime divisors of the order of M

[17]. Similarly, there is a class of M, denoted by $p+$, for each such prime with Thompson series fixed by $\Gamma_0(p)+$.

It is natural to interprete the Thompson series $T_g(\tau)$ as a contribution to the partition function for a further orbifolding of $\mathcal{V}^\natural$ with respect to g [18,14]. In particular, we expect that under $\tau \to -1/\tau$, $T_g(\tau)$ transforms to the partition function for a g twisted sector $\mathcal{V}_g$ as follows:

$$T_g(\tau) = g \underset{1}{\square}^{\natural} \to 1 \underset{g}{\square}^{\natural} = N_g q^{E_0^g} + \ldots \tag{7}$$

where $\natural$ denotes a trace contribution to the orbifolding of $\mathcal{V}^\natural$ and $\mathcal{V}_g$ has vacuum energy E_0^g and degeneracy N_g. For many classes of M, the method of construction of $\mathcal{V}_g$ is not known. However, for certain elements discussed below and some others, a construction can be given [14,13].

Consider now this orbifold picture of $T_g(\tau)$ for the prime classes $p+$ and $p-$, although the analysis given can be generalised to all classes [14,19,13]. Under a modular transformation $\gamma : \tau \to \frac{a\tau+b}{c\tau+d}$ we find $g \underset{1}{\square}^{\natural} \to g^{-d} \underset{g^c}{\square}^{\natural}$ assuming that no extra global phasé occurs [20] (such a phase corresponds to $h \neq 1$ in the original Moonshine conjectures [14,13]). For $\gamma \in \Gamma_0(p)$ with $c = 0 \bmod p$ we find $\gamma : T_g(\tau) \to T_{g^{-d}}(\tau) = T_g(\tau)$ since d and p are relatively prime and $T_g(\tau)$ is $\Gamma_0(p)$ invariant.

The genus zero property can be also understood as follows. $T_g(\tau)$ always has a simple pole at $q = 0$ ($\tau = \infty$). The only other possible pole for $T_g(\tau)$ is at $\tau = 0$ since the fundamental region $\mathcal{F}_p = H/\Gamma_0(p)$ for $\Gamma_0(p)$ has only these two cusp points [21]. From (7), $T_g(\tau)$ has a pole at $\tau = 0$ if and only if $E_g^0 < 0$. Thus $T_g(\tau)$ is a hauptmodul for $\Gamma_0(p)$ if and only if $E_g^0 \geq 0$. Also from (7), $T_g(W_p(\tau)) = 1 \underset{g}{\square}^{\natural}(p\tau)$, so that $T_g(\tau)$ is a hauptmodul for $\Gamma_0(p)+$ if and only if $E_g^0 = -1/p$ and $N_g = 1$.

For classes of type $p+$, $T_g(\tau) = 1 \underset{g}{\square}^{\natural}(p\tau)$ is a series in q with non-negative coefficients since the RHS of (7) is the $\mathcal{V}_g$ partition function. For classes of type $p-$, $T_g(\tau)$ has coefficients of mixed sign. In general, all classes of M can be divided into two such types i.e. classes with Thompson series invariant (or not invariant) under a Fricke involution $W_N : \tau \to -1/N\tau$ which are called Fricke (or non-Fricke) classes. There are a total of

121 Fricke classes all of which have non-negative coefficient Thompson series and 51 non-Fricke classes with mixed sign coefficients for similar reasons to the prime ordered classes described. This division of the classes of M will be important below.

The FLM Uniqueness Conjecture. FLM have conjectured that $\mathcal{V}^\natural$ is characterised (up to isomorphism) as follows [1]: *$\mathcal{V}^\natural$ is the unique meromorphic conformal field theory with modular invariant partition function $J(\tau)$ and central charge 24.* This is analogous to the uniqueness property of the Leech lattice as being the only even self-dual lattice in 24 dimensions without roots.

Let us now consider orbifold models based on other automorphisms a of the untwisted Leech lattice theory $\mathcal{V}^\Lambda$ lifted from automorphisms $\bar{a} \in \mathrm{Co}_0$ [19,22]. $\bar{a}$ will be chosen so that each model contains no massless operators, has a meromorphic OPA and is modular invariant with partition function $J(\tau)$ and hence, should reproduce $\mathcal{V}^\natural$. Each $\bar{a} \in \mathrm{Co}_0$ can be parameterised as follows

$$\det(x-\bar{a}) = \prod_{k|n}(x^k-1)^{a_k} \tag{8a}$$

$$\sum_{k|n} a_k = 0 \tag{8b}$$

with $\sum_{k|n} k a_k = 24$ where $k|n$ denotes that k divides n, the order of $\bar{a}$ and $\{a_k\}$ are integers. (8b) is imposed to ensure the absence of fixed points for $\bar{a}$ so that no massless operators in $\mathcal{V}^\Lambda$ survive the $\mathcal{P}_a$ projection. For $n=p$ prime, we have $a_p=-a_1=2d$ where $(p-1)2d=24$.

Since a is an OPA automorphism for $\mathcal{V}^\Lambda$, the a invariant subspace $\mathcal{P}_a\mathcal{V}^\Lambda$ also forms a closed meromorphic OPA. The partition function $\mathrm{Tr}_{\mathcal{P}_a\mathcal{V}^\Lambda}(q^{L_0})$ is not modular invariant, as before, necessitating the introduction of sectors $\mathcal{V}_a$ twisted by a. Thus under $\tau \to -1/\tau$

$$a\underset{1}{\square} = \frac{1}{\eta_{\bar{a}}(\tau)} \to 1\underset{a}{\square} = D_a^{1/2}\prod_{k|n}\eta(\tau/k)^{-a_k} = D_a^{1/2}q^{E_0^a}(1+O(q^{1/n})) \tag{9}$$

with $\eta_{\bar{a}}(\tau)=\prod_k \eta(k\tau)^{a_k}$ and $D_a=\det(1-\bar{a})$ where $D_a^{1/2}$ and $E_0^a=-\frac{1}{24}\sum_k \frac{a_k}{k}$ are the degeneracy and energy of the a twisted vacuum. Under $\tau\to\tau+n$, the a twisted partition function is invariant up to a phase $\exp(2\pi i n E_0^a)$. For modular consistency of the orbifold partition function we must have $nE_0^a = 0 \bmod 1$ i.e. there is no global phase anomaly [20].

In addition, if $E_0^a > 0$, then the a twisted sector does not reintroduce massless states. We therefore restrict ourselves to $\overline{a} \in \mathrm{Co}_0$ obeying [19]

$$\sum_{k|n} a_k = 0 \tag{10a}$$

$$E_0^a > 0 \tag{10b}$$

$$nE_0^a = 0 \bmod 1 \tag{10c}$$

There are a total of 38 classes of Co_0 [23] that obey these constraints [19]. If we relax condition (10c) then a further 13 classes of Co_0 obey only (10a-b) [24,13]. Each of these 13 classes is characterised by some $h \neq 1$ where $h|24$ with $h|k$ for all $a_k \neq 0$. In all 51 cases the parameters $\{a_k\}$ obey $a_k = -a_{nh/k}$ and so $E_0^a = 1/nh$ which violates (10c) for $h \neq 1$. $a \underset{1}{\square}$ is invariant up to phases of order h under $\Gamma_0(n)$ and is a hauptmodul for Γ_a with $\Gamma_0(N) \subseteq \Gamma_a \subset \mathcal{N}(N)$, $N = nh$, where Γ_a is one of the genus zero modular groups considered by Conway and Norton. Furthermore, since $E_0^a > 0$, $a \underset{1}{\square}$ cannot be Fricke invariant and hence these 51 hauptmoduls are the 51 non-Fricke Monster group hauptmoduls. Thus there is a correspondence between 51 classes $\{\overline{a}\}$ of Co_0 and the 51 non-Fricke classes of $\mathbb{M}$. We will explicitly identify an element $g_n \in \mathbb{M}$ of each such class below.

$\mathcal{V}_a$ with the partition function $1 \underset{a}{\square}$ of (9) has a standard construction [25]. Likewise, $\mathcal{V}_{a^k}$ twisted sectors must be introduced for modular invariance and OPA closure. Then the following intertwining non-meromorphic OPA should hold (schematically)

$$\psi_{a^j} \psi_{a^k} \sim \psi_{a^{j+k}} \tag{11}$$

with $\psi_{a^k}(z) \in \mathcal{V}_{a^k}$. Apart from the original Z_2 case, this OPA has only been rigorously constructed in the prime ordered cases [22]. We will assume that it is true in general. We therefore enlarge $\mathcal{V}^{\Lambda}$ by the introduction of $\mathcal{V}_{a^k}$ to $\mathcal{V}' = \mathcal{V}^{\Lambda} \oplus \mathcal{V}_a \oplus \ldots \oplus \mathcal{V}_{a^{n-1}}$ which forms a closed non-meromorphic OPA. The projection $\mathcal{V}_{\mathrm{orb}}^a = \mathcal{P}_a \mathcal{V}'$ then forms a meromorphic OPA. (10c) is sufficient to guarantee the modular invariance of the corresponding partition function. (10b) can be also shown to be sufficient to ensure no massless operators appear in $\mathcal{P}_a \mathcal{V}_{a^k}$ [19,13]. Thus, for the 38 automorphisms obeying (10a-c), the partition function is modular invariant and is given by $Z_{\mathrm{orb}}(\tau) = J(\tau)$. Therefore $\mathcal{V}_{\mathrm{orb}}^a \equiv \mathcal{V}^{\natural}$ according to the FLM uniqueness conjecture. Let us now consider some evidence to support this.

Let $M^a_{\rm orb}$ be the automorphism group of the OPA for $\mathcal{V}^a_{\rm orb}$ where we expect $M = M^a_{\rm orb}$. We define $i_a \in M^a_{\rm orb}$ of order n (which generalises the involution i in the original FLM construction) under which all the operators of $\mathcal{V}_{a^k}$ are eigenstates with eigenvalue $e^{2\pi ik/n}$. i_a is also an automorphism of $\mathcal{V}'$ and is 'dual' to the automorphism a where $\mathcal{P}_a\mathcal{V}' = \mathcal{V}^a_{\rm orb}$ and $\mathcal{P}_{i_a}\mathcal{V}' = \mathcal{V}^\Lambda$. Furthermore, we may reorbifold $\mathcal{V}^a_{\rm orb}$ with respect to i_a to reproduce $\mathcal{V}^\Lambda$ as before [13]

$$\begin{tikzcd} & \mathcal{V}' \arrow[dl, "\mathcal{P}_{i_a}"'] \arrow[dr, "\mathcal{P}_a"] & \\ \mathcal{V}^\Lambda \arrow[rr, shift left, "a"] & & \mathcal{V}^a_{\rm orb} \arrow[ll, shift left, "i_a"] \end{tikzcd} \tag{12}$$

Thus if $\mathcal{V}^a_{\rm orb} \equiv \mathcal{V}^\natural$, we can explicitly construct the twisted sectors $\mathcal{V}_{i_a^k}$ assumed earlier for $i_a \in M$. We may also compute the Thompson series for $i_a \in M^a_{\rm orb}$ by taking the trace over $\mathcal{V}^a_{\rm orb}$ to obtain

$$T^{\rm orb}_{i_a}(\tau) = {\rm Tr}_{\mathcal{V}^a_{\rm orb}}(i_a q^{L_0}) = \frac{1}{\eta_{\overline{a}}(\tau)} - a_1 \tag{13}$$

which is the hauptmodul for the genus zero modular group Γ_a introduced earlier [19]. Thus each $i_a \in M^a_{\rm orb}$ dual to a has the same Thompson series as a corresponding non-Fricke element of M e.g. for $\overline{a}$ of prime order p, $T^{\rm orb}_{i_a}(\tau) = [\eta(\tau)/\eta(p\tau)]^{2d} + 2d = T_{p-}(\tau)$. Note also, from (7), that $\mathcal{V}_{i_a}$ has vacuum energy $E_0^{i_a} = 0$ and degeneracy $-a_1 > 0$. (13) may be generalised to the other 13 classes violating (10c) where $\overline{a}^h$, of order $n' = n/h$, can be employed to construct an orbifold with partition function $J(\tau)$. Let g_n denote the lifting of $\overline{a}$ where $g_n^h = i_{a^h}$ is dual to a^h, a lifting of $\overline{a}^h$ (for $h = 1$, $g_n = i_a$). We may then compute the Thompson series for g_n as a trace over $\mathcal{V}^{a^h}_{\rm orb}$ to show that (13) again holds so that g_n has the same Thompson series as a non-Fricke element of M [13].

We may also compute the centraliser $C(g_n|M^{a^h}_{\rm orb}) = \{g \in M^{a^h}_{\rm orb} | g_n^{-1} g g_n = g\}$. For the 38 classes with $h = 1$ this consists of automorphisms that do not mix the sectors $\mathcal{P}_a\mathcal{V}_{a^k}$. For the other 13 automorphisms g_n, $C(g_n|M^{a^h}_{\rm orb}) \subset C(i_{a^h}|M^{a^h}_{\rm orb})$. In general, $c \in C(i_a|M^a_{\rm orb})$ must commute with a and therefore c is lifted from the automorphism $\overline{c} \in G_n = C(\overline{a}|{\rm Co}_0)/ < \overline{a} >$. One can then show that [24,13]

$$C(g_n|M^{a^h}_{\rm orb}) = \hat{L}_{\overline{a}}.G_n \tag{14}$$

where $\hat{L}_{\overline{a}} = n.L_{\overline{a}}$, an extension of $L_{\overline{a}} = \Lambda/(1-\overline{a})\Lambda$ by a cyclic group of order n. $\hat{L}_{\overline{a}}$ arises from the vaccum structure of $\mathcal{V}_a$ where $D_a = |L_{\overline{a}}|$. With $M^a_{\rm orb} = M$, (14) generalises a a well-known observation of Conway and Norton concerning the 5 prime classes where

$C(p-|M) = p_+^{1+2d}.G_p$ and $i_a = p-$ [15]. For the other 46 classes, there are 11 cases for which (14) can be checked using the available information about these centralisers [15,26]. In general, the order of these groups agrees with (14) in each case supporting the very likely validity of the result.

Both (13) and (14) support the conjecture that $\mathcal{V}^a_{\text{orb}} \equiv \mathcal{V}^\natural$. This can only be proved by finding a generalised version of σ in the FLM construction which mixes the untwisted and twisted sectors [1,12] i.e. there should exist some permutation group Σ_n which mixes the sectors of $\mathcal{V}^a_{\text{orb}}$ where $C(g_n|M)$ and Σ_n generate M. In the prime cases $p \neq 2$, Σ_p has been recently constructed and it has been rigorously shown that $M^a_{\text{orb}} = M$ for $p = 3$ and almost so for $p = 5, 7, 13$ [22].

Monstrous Moonshine from the Uniqueness of $\mathcal{V}^\natural$. Let us now assume that the FLM Uniqueness conjecture is correct. We can then argue that *Thompson series are hauptmoduls if and only if orbifolding $\mathcal{V}^\natural$ with respect to elements of M reproduces $\mathcal{V}^\natural$ or $\mathcal{V}^\Lambda$*. Thus Monstrous Moonshine is intimately linked to the uniqueness of $\mathcal{V}^\natural$.

From (12), orbifolding $\mathcal{V}^\natural$ with respect to the 38 non-Fricke elements i_a dual to a reproduces $\mathcal{V}^\Lambda$. We may similarly consider the orbifolding of $\mathcal{V}^\natural$ with respect to the Fricke elements $\{f\}$ with $h = 1$ which lead to a modular invariant theory $\mathcal{V}^f_{\text{orb}}$ [14,13], given that the operators $\mathcal{V}_{f^k}$ can be constructed. Assuming that the Thompson series are hauptmoduls we find that $\mathcal{V}^f_{\text{orb}} \equiv \mathcal{V}^\natural$ i.e. orbifolding $\mathcal{V}^\natural$ with respect to a Fricke automorphism reproduces $\mathcal{V}^\natural$ again. Thus we have [13]

$$\mathcal{V}^\Lambda \underset{i_a}{\overset{a}{\rightleftarrows}} \mathcal{V}^\natural \overset{f}{\longleftrightarrow} \mathcal{V}^\natural \tag{15}$$

For example, consider f an element of a prime class $p+$. Fricke invariance implies $1 \, \Box_{f^k}^{\,1} =$ $T_f(\tau/p) = q^{-1/p} + 0 + O(q^{1/p})$ so that there is a 'gap' in the spectrum of $\mathcal{V}_{f^k}$ and no massless operators are reintroduced in orbifolding $\mathcal{V}^\natural$. Thus the modular invariant partition function for $\mathcal{V}^f_{\text{orb}}$ is $J(\tau)$ and hence $\mathcal{V}^f_{\text{orb}} \equiv \mathcal{V}^\natural$. A similar argument can be made in the general case [13].

The converse to the above also holds i.e. assuming that (15) is true for all automorphisms of M that define a modular consistent theory, then the Thompson series are hauptmoduls. To see this, firstly consider an orbifolding with respect to $i_a \in M$ which reproduces $\mathcal{V}^\Lambda$. i_a must be dual to one of the 38 automorphisms obeying (10a-c) and has non-Fricke invariant Thompson series (13) which is the hauptmodul for a genus zero group.

Similarly, as discussed above, the other non-Fricke automorphisms can also be found with a corresponding genus zero Thompson series. For the remaining Fricke classes of M we provide an argument for f an element of prime order. We wish to show that $\mathcal{V}_f$ has the correct vacuum structure so that $T_f(\tau)$ is a hauptmodul for $\Gamma_0(p)+$. In the orbifolding of $\mathcal{V}^\natural$ with respect to f which reproduces $\mathcal{V}^\natural$, let $i_f \in M$ be dual to f with eigenvectors $\mathcal{V}_{f^k}$ for eigenvalue $e^{2\pi ik/n}$. Then it can be shown that $T_{i_f}(\tau) = T_f(\tau)$ so that i_f is in the same class as f. Furthermore, the centralisers obey $C(f|M) \subseteq C(i_f|M)$ with the necessary equality only when the $\mathcal{V}_f$ vacuum is unique i.e. $N_f = 1$. Since the twisted sector $\mathcal{V}_f$ does not reintroduce massless operators, the vacuum energy obeys either (a) $E_f^0 = -1/p$ or (b) $E_f^0 > 0$. (a) is possible because the absence of massless operators in $\mathcal{V}^\natural$ allows for a similar 'gap' in the spectrum of $\mathcal{V}_g$. If (b) holds, then $T_f(\tau)$ has a unique simple pole at $q = 0$ and must be a hauptmodul for $\Gamma_0(p)$ with $(p-1)|24$ and $T_f(\tau) = [\eta(\tau)/\eta(p\tau)]^{2d} + 2d$. However, this is impossible since then $E_f^0 = 0$ with $N_f = 2d$ from (7). Thus (15) implies that $\mathcal{V}_f$ has vacuum structure $E_f^0 = -1/f$ with $N_f = 1$ and hence, as described before, $T_f(\tau)$ is a hauptmodul for the genus zero group $\Gamma_0(p)+$ and f is of class $p+$. A similar argument can be given for the other Fricke classes [13].

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